Bare cavity study of the unstable Bessel resonator with finite apertures for the generation of Bessel-like beams

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A papá y mamá
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A detailed study of the unstable Bessel resonator is presented. The design of this cavity consists of a reflective axicon and a convex spherical output coupler. A matrix method analysis for the bare resonator is employed to extract the eigenmodes and eigenvalues of the cavity, which allows the retrieval of the fundamental and higher-order modes. Diffractive losses and relative phase shift behavior, in terms of both the varying radius of curvature of the output coupler and the cavity length, are studied with the matrix method and the Fox-Li algorithm. Direct comparison of the transverse mode profiles with a similar resonator employing a concave output coupler is performed, showing excellent agreement for large values of the radius of curvature. The effects of varying the aperture of the output coupler and the wedge angle of the axicon, on the transverse mode profiles and diffractive losses, are also considered.
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Chapter 1

Introduction

The science of optical resonators dates far back to 1958, when the works of Dicke [1] and Schawlow and Townes [2] were published. Since then an enormous amount of resonator configurations have been proposed for the generation of coherent radiation. The importance of the laser resonator relies in the fact that it is here where the laser beam is formed, furthermore it is the resonator geometry that determines, to a large extent, the shape characteristics of the beam generated by the laser system.

Resonator theory in its present form appeared only after the seminal work on laser resonators by Fox and Li [3]. This work demonstrated for the first time the existence of eigenmodes in open resonators, and pointed out the main properties of these modes for a few configurations. The work by Fox-Li was followed by the more general analysis of Boyd, Gordon and Kogelnik [4, 5], whose studies were mainly concerned about resonators consisting of two spherical end mirrors with arbitrary radii of curvature, and proposed a classification according to their diffraction losses. In more recent years, laser resonators have been employed for the generation of special types of beams commonly referred to as nondiffracting beams [6–9], particularly for the generation of Bessel beams.

Bessel light beams have been intensively studied since the publication of the foundational paper on lowest-order Bessel beams by Durnin et al., in 1987 [8, 9]. Special interest in these beams is due to their useful properties of suppressed diffraction divergence and their ability to reconstruct their intensity profiles during propagation. The potential applications for Bessel beams in optical alignment, optical interconnections, atom trapping [10, 11], and in high-resolution medical imaging [12] make them relevant for research, even though Bessel Beams constitute by themselves an interesting an suitable area of study. The realization of unlimited Bessel beams, which have an infinite extent and possess infinite energy, is impossible
in practice. However, in the real world it is possible to generate finite approximations to them.

In this thesis a rigorous analysis of the axicon-based unstable Bessel resonator (UBR) is presented, this resonator is intended for the generation of Bessel-like beams as its name indicates. The configuration considered shows two important modifications with respect to the conventional stable Bessel resonator (SBR) studied by Gutiérrez-Vega et al. [13], namely the output mirror is now convex-spherical, and the aperture ratio between the output coupler and the conical mirror has been reduced in favor to the output mirror. This work contributes to the science of laser resonators by introducing a formal analysis for the study of the UBR with finite apertures, and also by demonstrating for the first time the existence of higher-order Bessel modes resonating within the UBR. This research further extends the analysis performed by Tsangaris et al. [14], whose work was restricted to the examination of the fundamental mode of the cavity. In addition, this thesis presents an analysis of the diffraction losses in terms of the resonator parameters.

This thesis is structured as follows, in chapter 2, the general theory of Bessel beams and axicons is introduced. A complete geometrical description of the resonator is provided in chapter 3, where the resonator design is discussed, and the stability conditions are defined. Chapter 4 presents a formal wave optics treatment, now considering the effects of diffraction, besides the self-consistency integral equation is deduced. In chapter 5, the numerical techniques to solve the self-consistency equation are developed, including an advantageous matrix method that can extract the lowest \( N \) modes of the cavity, as well as the classical Fox-Li iterative algorithm useful to retrieve the resonator’s dominant mode.

In chapter 6, the effects of varying the curvature of the output mirror and the cavity length on the output transverse profile, the loss, and the resonance frequency shift of the lower modes are extensively studied. The analysis reveal that both the spherical mirror and the cavity length can be modified to minimize the losses due to diffraction, and to adjust the radial modulation of the Bessel rings of the output field. As seen in results, the transition between the SBR and the UBR is characterized by a continuous change in the associated diffraction losses and frequency shifts. Finally in chapter 7, the conclusions present a summary of the most relevant results, and reassert the fact that Bessel beams can be efficiently generated by means of the UBR.
Chapter 2

Bessel Beams and Axicons

2.1 Wave Equation

In the wave optics theory, the propagation behavior of light waves is mathematically described by the wave equation, which for free-space propagation is given by

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E(r,t) = 0. \]  

where \( c \) is the speed of light in vacuum. Each solution to Eq. (2.1) is a scalar function called optical field. In addition to reflection, refraction, and propagation, all optical fields are subject to diffraction and interference phenomena.

Diffraction is a natural phenomenon affecting the propagation of all classical wave fields, it is the responsible of the beam divergence of a gaussian beam, and of the intermission of light in the shadow region.

In 1987, Durnin [8] pointed out the existence of nondiffracting, or propagation invariant, solutions to Eq. (2.1) for scalar fields propagating into the source-free region \( z \geq 0 \). These particular solutions have the form

\[ E(x, y, z \geq 0, t) = \exp[i(\beta z - \omega t)] \int_0^{2\pi} A(\phi) \exp[i\alpha(x \cos \phi + y \sin \phi)] d\phi \]  

where \( A(\phi) \) is in general an arbitrary complex amplitude function, and \( \beta^2 + \alpha^2 = (w/c)^2 \). For the case that \( \beta \) is real Eq. (2.2) constitutes a nondiffracting solution of the wave equation. The term nondiffracting is somewhat unfortunate, since some authors prefer to name this type of solutions as invariant optical fields (IOF), due to the fact that all optical beams are subject to diffraction, however the term adopted for the matters of this thesis is nondiffracting. The
meaning of nondiffracting is understood in the sense that the field amplitude \( E(x, y, z, t) \) has the property

\[
E(x, y, z, t) = \exp[i(\beta z - \omega t)]E(x, y, z = 0, t = 0)
\]  

so when the time-averaged intensity is evaluated it satisfies

\[
I(x, y, z, t) = |E(x, y, z, t)|^2 = |E(x, y, z = 0, t = 0)|^2
\]  

this means that the intensity distribution of the beam, propagating in the \( z \) direction, is preserved across any plane orthogonal to the direction of propagation. If a picture of the transverse intensity distribution of the initial field at \( z = 0 \) and \( t = 0 \) is taken, and compared with a picture taken at a time \( t = t_1 \) after the beam has propagated a distance \( z = d \), the distributions should be the same, although the field itself should not.

2.2 Bessel beams

The simplest nondiffracting solution to Eq. (2.1) corresponds to the case when the amplitude function \( A(\phi) \) in Eq. (2.2) is independent of \( \phi \), in such case the field presents axial symmetry and its transverse disturbance distribution has the form of a zeroth order Bessel function, mathematically this is

\[
E(x, y, z, t) = \exp[i(\beta z - \omega t)] \int_0^{2\pi} \exp[i\alpha(x \cos \phi + y \sin \phi)] \frac{d\phi}{2\pi} = \exp[i(\beta z - \omega t)]J_0(\alpha r)
\]  

where \( r^2 = x^2 + y^2 \) and both \( \alpha \) and \( \beta \) are real constants, so that the field satisfies Eq. (2.4). For \( \alpha = 0 \) the solution reduces to the case of a plane wave propagating in the \( z \) direction, but for values in the range \( 0 < \alpha \leq 2\pi/\lambda \) the solution corresponds to a nondiffracting beam propagating in the \( z \) direction, whose transverse intensity distribution is given by

\[
I(r, z, t) = |E(r, z, t)|^2 = J_0^2(\alpha r)
\]

Figure 2.1 shows the plots for the \( |J_0(\alpha r)| \) and \( J_0^2(\alpha r) \) functions. It can be easily noted that a theoretical \( J_0 \) beam requires an infinite aperture. It is well known that the amplitude
of the oscillating Bessel function \( J_0(r) \) tends to decrease according to \( 1/\sqrt{r} \) when \( r \) goes to infinity [15], as depicted in Fig. 2.1 by the dashed line, therefore the intensity distribution is not square integrable. The latter means that, even though the beam intensity is concentrated around the optical axis, there exists an infinite number of side lobes supporting the main beam, each possessing an approximately equal amount of energy to that contained in the central spot. In experiment, an infinite amount of energy would be required to produce such beam, therefore it is only possible to generate finite approximations to nondiffracting beams by producing a limited number of these lobes over a finite plane, thus limiting the transverse extent and length of the nondiffracting beam, which makes it again subject to diffraction.

### 2.3 Bessel-Gauss beams

An alternative solution for the production of physically meaningful nondiffracting Bessel beams is to modulate the Bessel profile by an amplitude distribution with a more rapidly decaying rate, such modulated beams are known as apertured Bessel beams [16], and have the form

\[
F(r, 0) = A(r)J_0(\alpha r)
\]

for the particular case when the modulating aperture function \( A(r) \) corresponds to a gaussian envelope the resulting beam is termed a Bessel-Gauss beam, and Eq. (2.7) takes the form

\[
F(r, 0) = \exp \left( -\frac{r^2}{w^2} \right) J_0(\alpha r)
\]
where \( w \) is the waist of the gaussian term, and the temporal dependence \((-i\omega t)\) has been dropped for the sake of brevity. The radial modulation of the Bessel profile is readily evident from Fig. 2.2(a). The amplitude of the side lobes of the field in (a) rapidly decreases as \( r \) increases; the dashed curve corresponds to the factor \( \exp(-r^2/w^2) \), which serves as the “aperture” function mentioned before. This modulation has a stronger effect on the squared modulus of the Bessel-Gauss beam, as is shown in Fig. 2.2(b).

Mathematically, the function \( F(r, z) \) has a finite norm, as expected, which means that the beam carries a finite energy through any transverse plane, therefore it is physically realizable to a good extent.

### 2.4 Generation of Bessel beams

After Durnin’s experimental demonstration of the generation of Bessel beams by an annular slit, several methods for the generation of these beams have been proposed. Although Durnin’s experiment was not the first to produce beams with focal lines, McLeod [17] had demonstrated before that an axicon can produce this type of beams, he was the first to recognize them as nondiffracting.

In recent years, several ways to produce Bessel beams have been demonstrated, for instance, passive optical systems fed by laser light using a ring aperture and a positive lens [8,9], refractive or diffractive axicons [18,19], holographic methods [20,21], Fabry-Perot interferometers [22], or diffractive phase elements [23]. Active schemes to produce Bessel-type modes
in laser resonators have also been proposed, for example, arrangements based on annular intracavity elements [24], output mirrors with annular apertures to produce conical fields [25], graded-phase mirrors [7], and diode-pumped Nd:YAG lasers [26].

Axicon-based resonators supporting Bessel beams were proposed independently by Rogel et al. [27] and Khilo et al. [28] in 2001. This configuration has the advantage that it does not require intracavity optics or special shapes of the active medium. In 2003, Gutiérrez-Vega et al. [13] continued the exploration of the axicon-based resonator properties, extending their analysis to consider concave spherical output mirrors and developing a formal geometrical and wave analysis of the performance of the bare cavity. The axicon-based resonator with convex output coupler operating in the unstable regime was proposed by Tsangaris et al. [14] in 2003, however the scope of this initial work was restricted to the description of the output shape of the dominant Bessel mode using the classical iterative Fox-Li algorithm [3, 29]. In this research, the generation of higher order Bessel beams by means of the unstable axicon-based resonator is also discussed, as well as their propagation behavior inside and outside the resonator.

2.5 Axicons

Among the many methods for generating Bessel beams the use of an axicon is highly efficient [30–32]. An axicon is an optical element that, in contrast to an ordinary lens that creates a point focus, produces focal lines along the optical axis [see Fig. 2.3(a)]. The most remarkable characteristic of the axicon is that it transforms and incident plane wave into a conical wave, from the geometrical point of view this means that the trajectory of an incident ray, excepting a ray passing through the center of the axicon, is always changed by a constant angle $\theta_0$ towards the optical axis, as depicted in Fig. 2.3(b). There exist different types of axicons, for instance refractive or reflective axicons, diffractive axicons, and lens axicons [31], all of them have the same properties.

The conical waves produced by the axicon propagate with the same transverse component of the wave vector, and they are able to interfere. It is well documented [19,33,34] that the light distribution within the region where the conical waves superpose is a good approximation to a Bessel beam of the form given by Eq. (2.5), with $\alpha$ and $\beta$ playing the roles of the transverse and longitudinal components of the wave vector $k_T$ and $k_z$ respectively, this fact is shown in
Figure 2.3: (a) An axicon produces focal lines along the optical axis, (b) The trajectory of an input ray is deviated a constant angle $\theta_0$ towards the optical axis, (c) Generation of a Bessel beam by the superposition of the conical waves produced by the axicon.

Fig. 2.3(c); the incident plane wave is transformed into a conical wave after passing through the axicon, the shaded area represents the interference region where the superposition of the conical waves gives rise to the generation of the Bessel beam.

It is important to note that, depending on the coherent properties and angle of illumination of the incident light, the focal lines produced by the axicon will possess different characteristics [35, 36]. For the purposes of this work, only fully coherent light distributions will be considered and no restriction is imposed on the incidence angle of light.

The axicon is characterized by a linear phase radial transmittance given by

$$T(r) = \exp(-ik\theta_0 r)$$

(2.9)

where $k = 2\pi/\lambda$ is the wave number for free-space propagation, and $r$ is the radial coordinate.
The characteristic angle for a reflective axicon is just $\theta_0$, whereas for a thin refractive axicon with wedge angle $\alpha$ [see Fig. 2.3(b)] and refraction index $n$ is given by

$$\theta_0 = \arcsin(n \sin \alpha) - \alpha \simeq (n - 1)\alpha$$

(2.10)

as mentioned before, the refractive axicon was assumed to be thin, therefore the small angle approximation $\sin \alpha \simeq \alpha$ holds. In practice, there is no numerical distinction between a refractive or a reflective axicon, they share the same characteristics and produce the same results, but for the matters of resonator design, that are covered in the next chapter, a totally refractive axicon can not be equally employed in the same way as a reflective one.
Chapter 3

Geometrical analysis of the Unstable Bessel Resonator

3.1 Resonator design

The configuration of the UBR consists of a conical mirror with characteristic angle $\theta_0$ and a partially reflective convex-spherical output mirror separated a distance $L$, as depicted in Fig. 3.1(a). The conical mirror has the effect of transforming an incident plane wave into a converging conical wave, changing the trajectory of a horizontal input ray by a constant angle $2\theta_0$ towards the optical-axis [see Fig. 3.1(a)]. In practice, the reflective conical mirror can also be constructed using a refractive axicon with index of refraction $n$ and wedge angle $\alpha$; the base of the axicon is perfectly covered by a totally reflecting plane mirror, as shown in Fig. 3.1(b). In this case the conical angle $\theta_0$ is related to the axicon parameters according to Eq. (2.10). In conventional unstable resonators the convex output mirror is totally reflective, so that the laser output is taken as a diffraction-coupled beam passing around it, thus generating an annular beam; in contrast, the design of the UBR includes a partially reflective output mirror, which allows the transmission of a fraction of the incident radiation, therefore the beam in the UBR passes through the output coupler.

There are four parameters that fully characterize the UBR design, namely the characteristic angle $\theta_0$ of the conical mirror and its radius of aperture $a_1$, as well as the aperture radius of the output mirror $a_2$, and its radius of curvature $R$. These parameters completely define the properties and transverse extent of the generated laser beam. Another important parameter is the cavity length $L$, which in order to satisfy the requirement that the UBR reduces to the ideal stable bessel resonator (SBR) [27], when the spherical output mirror
Figure 3.1: Design of the resonator with (a) reflective and (b) refractive axicon. The convex spherical mirror is placed at a distance $L$ from the axicon.

becomes a flat mirror, is chosen to be

$$L = \frac{a_1}{2 \tan \theta_0} \simeq \frac{a_1}{2 \theta_0} = \frac{a_1}{2(n-1)\alpha},$$  \hfill (3.1)

it can clearly be seen the dependence of $L$ on the parameters of the conical mirror. On the other hand, the radius of curvature $R$ of the output mirror is a free parameter that can be adjusted to modify the diffraction properties of the cavity, and of its resonating modes.

Apart from the specification of the previous parameters, the aperture radii of the input and output mirrors play an important role on the diffraction properties of the cavity, in fact they have strong implications in the behavior of the diffraction losses and frequency phase shifts, as will be shown in detail in the following chapters. For now, suffice it to say that they can take different values, although it is possible to show that there exist one value that minimizes diffraction losses.

3.1.1 Considerations on the sign of $R$

Three important cases regarding the values of the radius of curvature $R$ of the output mirror can be considered, for instance the case $R > 0$ corresponds to a concave output coupler, for which the resonator is geometrically stable, and the field at the output plane can be approximated by an $l$-th order Bessel-Gauss beam [13]

$$u(r) = J_l(k_l r) \exp \left( -\frac{r^2}{w^2} \right) \exp \left[ i l \phi + \Phi \right],$$  \hfill (3.2)
where

\[
    w^2 = w_0^2 \left[ 1 + \left( \frac{L}{z_R} \right)^2 \right],
\]
\[
    w_0 = (2z_R/k)^{1/2},
\]
\[
    z_R = \left[ L (R - L) \right]^{1/2},
\]
\[
    \Phi = kR^2/2R
\]
\[
    k_t = k \sin \theta_0
\]

the waist of the modulating gaussian factor is denoted by \( w \), \( w_0 \) and \( z_R \) are the waist and Rayleigh range of the equivalent gaussian beam; the term \( \Phi \) is the phase of the spherical wavefront at the output mirror and \( k_t \) is the transverse wave number.

Eq. 3.2 is just a generalization of Eq. (2.8), but now including an angular phase variation of the form \( \exp \left[ i (l\phi + \Phi) \right] \), which implies that higher order Bessel-Gauss modes also possess angular momentum.

In the limit of a flat output coupler (\( R \to \infty \)), the resonator reduces to the case of the ideal SBR; the waist of the modulating gaussian amplitude goes to infinity, and the phase \( \Phi \) becomes zero, which means that the wavefront propagates as plane waves rather than spherical, therefore Bessel-Gauss modes reduce to Bessel beams of the form \( J_l(k_t \rho) \exp(i l \phi) \).

A not so fortunate situation occurs for the case \( R < 0 \), which corresponds to the UBR, since there is no analytical expression for the beam modes, however it is always possible to retrieve them numerically. Nonetheless, the beams produced by the UBR closely resemble Bessel beams. Anyhow, in real resonators the finite extent of the mirrors slightly modifies the transverse shape of the ideal Bessel modes such that, even in the case of having a plane output mirror, the resonating modes are modulated by a bell-shaped envelope.

Output fields with specific transverse features can be generated by appropriately choosing the geometric parameters of the UBR in Fig. 3.1. As it will be confirmed in the following sections, under the paraxial regime and neglecting the finite size of the apertures, the output field may be approximated by the product of an ideal Bessel beam and the field produced by a half-symmetric unstable resonator in which one mirror is planar (axicon plane) and the other convex (output plane). When considering the finite extent of the mirrors, the output function seems to be modulated by a near top-hat function, which is a good approximation
to the ideal situation of having a constant amplitude modulation. As occurs in the SBR, the radial separation of the Bessel rings in the UBR will depend only on the angle $\theta_0$ of the conical mirror.

### 3.2 Equivalent lens-guide system

A pure geometrical analysis provides a useful insight to the mode properties of the UBR. The conventional method for performing a geometrical and stability analysis, in the theory of optical resonators, is to represent the resonator in an equivalent system consisting of lenses and apertures [37]. The equivalent lens-guide system of the UBR is shown in Fig. 3.2, where the conical mirror is represented by a double refractive axicon, and the convex mirror by a divergent lens. This method is advantageous from the point of view that the conditions of stability, for an optical resonator, are found by a rather simple procedure of matrix multiplication. Each lens element is represented by a characteristic matrix, usually referred to as the ABCD matrix, constructed performing a paraxial ray analysis. The complete ABCD matrix of the lens-guide system is then formed by the multiplication of the corresponding matrices of all the constituting elements.

![Figure 3.2: Lens-guide equivalent system and self-reproducibility condition for ray trajectories after one and two round trips.](image_url)
3.3 Paraxial ray theory of ABCD systems

There are three elements composing the equivalent lens-guide system shown in Fig. 3.2, namely the double axicon, the divergent lens, and a free space segment between them, their ray transfer ABCD matrices are calculated by relating the displacement $r_1$ and slope $\theta_1$ of an incident ray at point $z_1$ to the displacement and slope at a point $z_2$, as depicted in Fig. 3.3. It should be emphasized that the analysis developed in this section assumes all considered rays to be paraxial, which means they are under the assumption of nearly unidirectional propagation along $z$, therefore the small angle approximation $\tan \theta_1 \approx \sin \theta_1 \approx \theta_1$ can be used. The simplest case corresponds to the propagation of a ray through a free-space segment of length $L$, from Fig. 3.3(a) it can be seen that the slope and position of a ray after traveling a distance $L$ are given by

$$r_2 = r_1 + L\theta_1,$$
$$\theta_2 = \theta_1$$

in matrix form

$$\begin{bmatrix} r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix},$$

$$\text{(3.3)}$$

Figure 3.3: Paraxial ray analysis of the equivalent lens-guide system composing elements. (a) Free space region of length $L$, (b) double axicon, (c) divergent lens.
the ABCD matrix is the matrix of coefficients describing the system, namely

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix}
\]  \hspace{1cm} (3.4)

A similar analysis can be applied to the double axicon shown in Fig. 3.3(b), giving the following system [13]

\[
\begin{align*}
    r_2 &= r_1, \\
    \theta_2 &= \theta_1 - 2\theta_0 = r_1 \left( \frac{-2\theta_0}{r_1} \right) + \theta_1
\end{align*}
\]  \hspace{1cm} (3.5)

which in its matrix representation takes the form

\[
\begin{bmatrix}
r_2 \\
\theta_2
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-2\theta_0/r_1 & 1
\end{bmatrix} \begin{bmatrix}
r_1 \\
\theta_1
\end{bmatrix},
\]  \hspace{1cm} (3.6)

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-2\theta_0/r_1 & 1
\end{bmatrix}
\]  \hspace{1cm} (3.7)

where \((r_1, \theta_1)\) and \((r_2, \theta_2)\) are the position and slope of the input and output rays, respectively. Note that the radius \(r\) is explicitly part of the ABCD matrix for the double axicon, this fact is remarkable since for conventional elements the ABCD matrix is independent of the initial displacement of an input ray, also it will eventually play a significant role defining the self-consistency condition of the resonator. The same previously outlined procedure is employed to determine the ABCD matrix for the divergent lens, and can be derived with the help of Fig. 3.3(c), the resulting ABCD matrix is then given by

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
2/R & 1
\end{bmatrix}
\]  \hspace{1cm} (3.8)

where \(R\) is the radius of curvature of the convex mirror, and the fact that \(f = R/2\) has been used. An important characteristic of ABCD matrices is that their determinant equals unity, namely \(\det(ABCD) = AD - BC = 1\), as can be verified in Eqs. (3.3), (3.7), and (3.8).

For the construction of the ray transfer matrix for the complete system is important to note that, in general, the matrix product \(M_1M_2\) is not the same as \(M_2M_1\), therefore the order
in which the multiplication of the ABCD matrices of the comprising elements is performed is important. Thus with the reference planes placed just before the double axicon, the ABCD matrix for the complete cavity is constructed from right to left following the propagation of a ray through each element inside the lens-guide system; the matrix of the first optical element (double axicon) is multiplied on the left by the ray matrix of the second element (free-space segment), multiplied on the left by the third element (divergent lens), and finally multiplied on the left by the matrix of the fourth element (free-space segment), resulting in

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
2/R & 1
\end{bmatrix}
\begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-2\theta_0/r_1 & 1
\end{bmatrix}
\]

\[= \begin{bmatrix}
1 + \frac{2L}{R} + \left(\frac{-2\theta_0}{r_1}\right)2L \left(1 + \frac{L}{R}\right) & 2L \left(1 + \frac{L}{R}\right) \\
\left(\frac{-2\theta_0}{r_1}\right) \left(1 + \frac{2L}{R}\right) + \frac{2}{R} & 1 + \frac{2L}{R}
\end{bmatrix}
\]

\[= \begin{bmatrix}
D + \left(\frac{-2\theta_0}{r_1}\right)B & B \\
C & D
\end{bmatrix}
\]

\[(3.9a) = (3.9b) = (3.9c)

\]

Note that \(AD - BC = 1\), since the product of matrices whose determinant is unity is also a matrix with unitary determinant.

### 3.4 Stability and self-reproducing trajectories

#### 3.4.1 Stability analysis

An important issue in the theory of resonators, is to determine whether it is stable or unstable, and to find the conditions that should be satisfied in order to achieve one of the two regimes, the stability nature of a resonator has strong implications on its diffraction losses behavior and modal discrimination; apart from stability it is desirable to explore the existence of self reproducing ray trajectories after one and two round trips [see Fig. 3.2], since they will indicate the existence of self reproducing field configurations where the field propagates at angles \(\theta_0\) and \(2\theta_0\). It is important to note that the ray trajectories in the resonator are folded, whereas in the lens-guide system are unfolded.

The unstable nature of the UBR in Fig. 3.1 comes from the inclusion of the convex output mirror, all rays reflecting upon its surface become more and more dispersed the further they

17
reflect on it. However, making use of the equivalent lens-guide system, it is possible to find stable trajectories for rays traveling within the resonator, this means that those rays will be refocusing periodically as they propagate. The criterion for stability (beam confinement) in the geometrical sense can be established from the ABCD matrix of the equivalent lens-guide system. The usual procedure yields the condition for stability [37]

\[-1 \leq \frac{1}{2}(A + D) \leq 1\]  

(3.10)

where \(A\) and \(D\) are the elements of the ABCD matrix for the complete system, thus substitution of their corresponding values from Eq. (3.9b) gives

\[-1 \leq \frac{1}{2} \left[ 2 \left(1 + \frac{2L}{R}\right) + \left(-\frac{2\theta_0}{r_1}\right)2L \left(1 + \frac{L}{R}\right) \right] \leq 1\]

simplification of the previous equation results in

\[\frac{1}{R + L} \leq \frac{\theta_0}{r_1} \leq \frac{1}{L}\]

which gives the two following conditions

\[r_1 \leq \theta_0(R + L)\] \hspace{1cm} (3.11)
\[r_1 \geq a/2\] \hspace{1cm} (3.12)

Equations (3.11) and (3.12) determine a stability range for rays traveling within the cavity, it is interesting to note that rays passing through the vertex of the axicon do not satisfy the stability condition, and will eventually escape from the resonator, causing energy losses that will be studied in the following sections.

### 3.4.2 Conditions for self reproducing trajectories

An eigenvalue analysis over the ABCD Matrix helps us to find the conditions for periodic trajectories that are self-reproducible after one and two round-trips respectively. The mathematical problem is that of finding the eigenvectors \((r, \theta)\) of the self-reproducibility equation
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
r \\
\theta
\end{bmatrix}
= \pm
\begin{bmatrix}
r \\
\theta
\end{bmatrix},
\]  
(3.13)

where the plus and minus signs correspond to trajectories that are self-reproducible after one and two round-trips respectively [see Fig. 3.2]. Nonzero solutions are only possible if

\[
\det
\begin{bmatrix}
A \mp 1 & B \\
C & D \mp 1
\end{bmatrix}
= 0.
\]  
(3.14)

Taking advantage of the fact that \(AD - BC = 1\), it can easily be noticed that Eq. (3.14) vanishes only if \(A + D = \pm 2\). The eigenvalues corresponding to self-reproducing trajectories after one and two round trips are obtained by substituting the values of \(A\) and \(D\), from Eq. (3.9b), into conditions \(A + D = \pm 2\), and solving for \(r\), yielding

\[
r_{\text{one r-t}} = (L + R) \theta_0, \quad r_{\text{two r-t}} = L \theta_0 = \frac{a_1}{2}.
\]  
(3.15)

As expected, the eigenangles \(\theta\) associated to each \(r\) are found to be \(\theta_{\text{one r-t}} = \theta_{\text{two r-t}} = \theta_0\). Note that \(r_{\text{two r-t}}\) depends only on the aperture of the axicon, so it is always present in the cavity, but the existence of one-round trip stable trajectories is not always guaranteed because it depends on the radius of curvature of the output mirror.

In Fig. 3.2 the typical two round-trips stable trajectory is shown. Evidently the rays have rotational symmetry, hence there is an eigencone of rays whose apex is placed at the center of the output mirror and its circular base intersects the axicon plane at radius \(r = a_2/2\). The graphical representation of one round-trip stability condition is also shown in Fig. 3.2. The eigenangle \(\theta_0\) is a restriction of the axicon, whereas the eigenradius \((L + R) \theta_0\) is determined primarily by the spherical surface. For the Bessel resonator with plane output mirror, it is also possible that other self reproducing trajectories after a larger number of round trips exist, for example field distributions traveling with angles \(3\theta_0\) or \(4\theta_0\) (known as multipass modes), but they require a larger radius on both input and output mirrors, therefore they will only be able to oscillate if the aperture radii of the end mirrors is large compared to the transverse extent of the field. If at least one of the apertures is smaller than \(2\theta_0L\), then multipass oscillation is not possible [28]; in the UBR only two self reproducing trajectories exist, since multipass
oscillation cannot occur due to the finite and different sizes of the apertures, and to the large
diffraction losses induced by the cavity.

In conclusion, Fig. 3.2 and Eq. (3.15) tell us that every plane wave component of a
resonating Bessel mode in the UBR must travel at an angle of $2\theta_0$ towards the optical axis,
which is consistent with the fact that a conical mirror of characteristic angle $\theta_0$ will transform
and incident plane wave into a Bessel beam, with conicity angle $2\theta_0$. 
Chapter 4

Wave optics analysis

The geometrical description provides a useful but limited approximation to the mode properties of the UBR. A more detailed understanding of the mode distribution is obtained by performing a wave optics analysis of the resonator, in which diffraction effects are considered, and the propagation of light wave fields is described by means of the Huygens-Fresnel integral.

4.1 Diffraction integral

Scalar diffraction theory, where light is considered to be a wave, is based on the diffraction integral for free space

\[
u(x, y, z) = \frac{1}{i\lambda} \int \int u_0(x_0, y_0, 0) \exp[ikR(x, x_0, y, y_0, z)] \frac{R(x, x_0, y, y_0, z)}{R(x, x_0, y, y_0, z)} dxdy_0,
\]

which describes the propagation of a light wave between two different planes with coordinates \((x_0, y_0, 0)\) in the reference, or aperture plane, and \((x, y, z)\) in the image plane. \(R(x, x_0, y, y_0, z)\) is the distance between two considered points in the image and aperture planes, and is given by

\[
R(x, x_0, y, y_0, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}
\]

it is also assumed that light is a monochromatic wave of wavelength \(\lambda\), and that the disturbance \(u(x, y, z)\) presents a temporal dependence of the form \(u(x, y, z, t) = \exp(-i\omega t)u(x, y, z)\) where \(\omega = 2\pi c/\lambda\) is the angular frequency and \(c\) the speed of light. The integration is performed over the entire aperture plane \((x_0, y_0, 0)\).

In case that an optical element is inserted into the aperture, the resulting field can be affected both in its amplitude and phase. Based on a thin element assumption, the optical
element can be described by a transmittance function

$$T(x_0, y_0) = A(x_0, y_0) \exp [ik\phi(x_0, y_0)],$$

(4.3)

with $A(x_0, y_0)$ and $\phi(x_0, y_0)$ being real functions of amplitude and phase transmission respectively. Mathematically, the incident field on the optical element is multiplied by the transmittance function, yielding the field distribution just after the aperture, so that the field at the image plane is calculated through the diffraction integral in Eq. (4.1), but now including the effect of the optical element, namely

$$u(x, y, z) = \frac{1}{i\lambda} \int \int u_0(x_0, y_0, 0) T(x_0, y_0) \frac{\exp [ikR(x, x_0, y, y_0, z)]}{R(x_0, y_0, z)} dx_0 dy_0,$$

(4.4)

In this thesis, the optical elements composing the UBR does not modify the amplitude of the incident field, which means they have a phase-only transmittance function.

### 4.1.1 Paraxial approximation

The use of the full diffraction integral can be considered too complicated for some applications, such as the analysis of the mode distribution in the UBR which concerns us now, depending on the geometry and on the required accuracy, there are alternative approximations that can be used with excellent results. The two most commonly employed approximations to Eq. (4.1) are the Huygens-Fresnel and Fraunhofer diffraction integrals. The Huygens-Fresnel integral is also the exact solution of the paraxial wave equation, and is fully adequate for describing nearly all optical resonator and beam propagation problems arising in the analysis of the UBR, and in general in all real laser systems. Consequently, in the Fresnel region the assumption that the distance to the image plane is large is used, i.e. $z \gg x_0, y_0, x, y$, and is valid as long as all light remains close to the optical axis. Within the Fresnel region, it is possible to use $R \approx z$ in the denominator of Eq. (4.4), since $R$ and $z$ have comparable sizes, however this approximation can not be equally employed inside the argument of the exponential function because the wave number $k$ has a large value compared to $R$, thus a small variation of $R$ will produce a significant change in the integrand. A more convenient approximation can be achieved employing the binomial expansion $\sqrt{1 + x} \approx 1 + x/2 - 3x^2/8 + \ldots$ of
Eq. (4.2), preserving only the quadratic term it becomes [29]

\[ R(x, x_0, y, y_0, z) \approx z + \frac{x^2 + y^2}{2z} + \frac{x_0^2 + y_0^2}{2z} - \frac{xx_0 + yy_0}{z}, \]  \hspace{1cm} (4.5)

substitution of Eq. (4.5) in Eq. (4.4) yields the Huygens-Fresnel diffraction integral in cartesian coordinates

\[
\begin{align*}
  u(x, y, z) &= \frac{\exp(ikz)}{i\lambda z} \int \int u_0(x_0, y_0, 0) T(x_0, y_0) \exp \left[ \frac{ik}{2z} (x^2 + y^2) \right] \\
  &\times \exp \left[ \frac{ik}{2z} (x_0^2 + y_0^2) \right] \exp \left[ -\frac{ik}{z} (xx_0 + yy_0) \right] dx_0 dy_0, \\
\end{align*}
\]  \hspace{1cm} (4.6)

Despite its accuracy, the Huygens-Fresnel integral is much easier to evaluate than the complete diffraction integral, and offers excellent result for the aim of this thesis. On the other hand, the Fraunhoffer diffraction integral is useful for longer propagation distances, when it is also possible to disregard the quadratic phase components, but since it does not offer advantages over the Fresnel integral concerning the analysis of the UBR, it is dismissed in this thesis, for a more complete treatment the reader may consult reference [38].

### 4.1.2 Huygens-Fresnel integral in cylindrical coordinates

For systems with cylindrical symmetry, as is the case of the UBR, it is convenient to write the integral 4.6 in cylindrical coordinates, leading to the result

\[
\begin{align*}
  u(r, \varphi, z) &= \frac{\exp(ikz)}{i\lambda z} \int \int u_0(r_0, \varphi_0, 0) T(r_0, \varphi_0) \exp \left[ \frac{ik}{2z} (r^2 + r_0^2) \right] \\
  &\times \exp \left[ -\frac{ik}{z} (rr_0 \cos(\varphi - \varphi_0)) \right] r_0 dr_0 d\varphi_0, \\
\end{align*}
\]  \hspace{1cm} (4.7)

assuming that the wave function possess \(l\)-th order azimuthal symmetry, which is reasonable given the cylindrical geometry of the resonator, then the field may be expressed as

\[ u(r, \varphi) = u_l(r) \exp(\pm il\varphi) \]  \hspace{1cm} (4.8)
for both $u$ and $u_0$, with this assumption Huygens-Fresnel’s integral can be written in the simpler form

$$u_l(r, z) = \frac{2\pi(-i)^{l+1}}{\lambda z} \int u_0(r_0, 0) T(r_0, \varphi_0) \exp \left[ \frac{ik}{2z} (r^2 + r_0^2) \right] J_l \left( \frac{k}{z} r r_0 \right) r_0 dr_0,$$

(4.9)

where $J_l$ is the $l$-th order Bessel function. The term $\exp(ikz)$ in Eq. (4.9) has been omitted since it is the transverse variation of $u_l(r)$ what primarily interest us.

### 4.2 Cascading propagation integrals

In the equivalent lens-guide system shown in Fig. 3.2, the complete round-trip inside the resonator can be broken into two segments. The first segment corresponds to the propagation from just before the double axicon ($RP_1$) to the output mirror ($RP_2$), and then to the original starting plane ($RP_3$).

The Huygens-Fresnel integral for the propagation of an azimuthal radial mode $u_{lp}$, of the mode field $u(r_m, \varphi) = \sum_{p=0}^{\infty} \sum_{l=-\infty}^{\infty} u_{lp}(r_m) \exp(il\varphi)$, through a paraxial cylindrical system from plane $RP_m$ [coordinates $(r_m, \varphi)$] to plane $RP_n$ [coordinates $(r_n, \varphi)$], is given by [29]

$$u_{lp}(r_n) = \int_0^{a_m} K_l(r_m, r_n) T_m(r_m) u_{lp}(r_m) dr_m,$$

(4.10)

where the azimuthal and radial mode indices are given by $l = 0, \pm 1, \pm 2, \ldots$ and $p = 1, 2, 3, \ldots$ respectively. The transmittance function of the optical element or aperture located at plane $RP_m$ is $T_m(r_m)$, and for this analysis only depends on the radial coordinate. The kernel $K_l(r_m, r_n)$ for the free-space propagation segment of length $L$, between the planes $m$ and $n$, is given by

$$K_l(r_m, r_n) = (-i)^{l+1} \left( \frac{k}{L} \right) r_m J_l \left( \frac{k}{L} r_m r_n \right) \exp \left[ \frac{ik}{2L} \left( r_m^2 + r_n^2 \right) \right],$$

(4.11)

with $J_l(\cdot)$ being the $l$-th order Bessel function. It can readily be seen that the kernel in Eq. (4.11) is exactly the same as the kernel of Eq. (4.9), therefore Eq. (4.10) is just a reduced form of Eq. (4.9). For the system shown in Fig. 3.2 the upper integration limit of the integral in Eq. (4.10) corresponds to the radius of the aperture at $RP_m$.

The double axicon and the convex output mirror can be characterized by their corre-
sponding transmittance functions

\[ T_1(r_1) = \exp(-i 2k \theta_0 r_1), \quad r_1 \leq a_1 \] \hspace{1cm} (4.12a) \]
\[ T_2(r_2) = \exp(-ik r_2^2 / R), \quad r_2 \leq a_2 \] \hspace{1cm} (4.12b) \]

where a time dependence \( \exp(-i \omega t) \) has been assumed, and irrelevant constant phase shifts have been ignored. Since the optical elements composing the UBR are phase-only elements, it is deduced from Eqs. (4.3) and (4.12) that their amplitude functions are equal to unity. Notice that the double axicon exhibits a linear phase radial variation instead of the quadratic variation of the spherical lens.

The round-trip propagation for a mode in the UBR is described by the following pair of coupled integral equations [39, 40]

\[ u_{lp}(r_2) = \int_{0}^{a_1} K_l(r_1, r_2) T_1(r_1) u_{lp}(r_1) \, dr_1, \] \hspace{1cm} (4.13) \]
\[ u_{lp}(r_3) = \int_{0}^{a_2} K_l(r_2, r_3) T_2(r_2) u_{lp}(r_2) \, dr_2. \] \hspace{1cm} (4.14) \]

By direct substitution of Eq. (4.13) into Eq. (4.14) the round-trip integral from \( RP_1 \) to \( RP_3 \) is found to be

\[ u_{lp}(r_3) = \int_{0}^{a_1} H_{l13}^{13}(r_1, r_3) u_{lp}(r_1) \, dr_1, \] \hspace{1cm} (4.15) \]

where the fact that radial coordinates in both planes are numerically the same was used, as illustrated in Fig.1(c).

The kernel in Eq. (4.15) includes information about both propagation kernels in Eqs. (4.13) and (4.14), and the transmittance functions \( T_1 \) and \( T_2 \), explicitly

\[ H_{l13}^{13}(r_1, r_3) = \int_{0}^{a_2} H_{l23}^{23}(r_2, r_3) H_{l12}^{12}(r_1, r_2) \, dr_2, \] \hspace{1cm} (4.16) \]

with \( H_{l12}^{12}(r_1, r_2) = K_l(r_1, r_2) T_1(r_1) \) and \( H_{l23}^{23}(r_2, r_3) = K_l(r_2, r_3) T_2(r_2) \), where the superscripts \((m, n)\) mean that the kernels are evaluated from \( RP_m \) to \( RP_n \).
4.3 Self-consistency integral equation

At the reference plane, each eigenmode \(u_{lp}\) in the cavity satisfies the self-consistency integral equation

\[
\gamma_{lp} u_{lp}(r_3) = \int_{0}^{a_1} H^{13}_l(r_1, r_3) u_{lp}(r_1) \, dr_1,
\]

(4.17)

The kernel \(H^{13}_l\) is complex symmetrical but not Hermitian, and therefore the eigenvalue \(\gamma_{lp}\) is complex. The non-hermitian nature of the self-consistency integral equation represents a major difficulty in the study of the transverse mode structure of the UBR, and in general of all open optical cavities, since the existence and completeness of a set of eigenfunctions cannot be guaranteed in advance. However, the existence of at least one nonzero eigenvalue has been proven theoretically by Hochstadt \[41\]; in experiment it is clear that, in the presence of a gain medium, steady-state oscillation can be achieved by at least the lowest loss mode, and that limiting apertures can help the structure achieve mode stability \[39, 42\]. Another important fact to be outlined is that the eigenfields are not power orthogonal in the usual sense, rather they are biorthogonal according to Siegman \[42\].

Fractional power loss and phase shift calculations for a particular mode \((l, p)\) can be carried out by means of the complex eigenvalue \(\gamma_{lp}\), which can be expressed in its polar form as

\[
\gamma_{lp} = |\gamma_{lp}| \exp \left(i\beta_{lp}\right),
\]

where the magnitude \(|\gamma_{lp}|\) defines the fractional power loss per transit

\[
\Gamma_{lp} = 1 - |\gamma_{lp}|^2,
\]

(4.18)

(4.19)

and the eigenangle \(\beta_{lp}\) gives the phase shift suffered by the mode in addition to the longitudinal phase shift \(kL\). The resonant condition requires that the total phase shift \(\Phi\) along the axis of the cavity be an entire multiple of \(\pi\) radians, thus

\[
\Phi = kL + \beta_{lp} = q\pi,
\]

(4.20)

where \(q\) is the number of half wavelengths of the axial standing wave pattern. The resonance frequency for a particular mode \((l, p)\) can be defined from Eq. (4.20) by replacing \(k = 2\pi\nu/c\)
and solving for $\nu$, giving

$$\nu_p = \nu_0 \left( q - \frac{\beta_p}{\pi} \right),$$

(4.21)

where $\nu_0 = c/2L$ is the fundamental beat frequency, i.e., the frequency spacing between successive longitudinal resonances.

The eigenvalue $\gamma_{01}$ which possesses the largest magnitude is directly associated to the fundamental mode of the resonator, corresponding to the eigenfunction $u_{01}$, and is actually the distribution presenting the lowest diffraction losses. It is shown in the following chapters that this fundamental mode corresponds to a zeroth order Bessel beam, and that higher order Bessel beams are also resonating modes of the UBR.
Chapter 5

Numerical considerations

The most desirable situation when solving Eq. (4.17) is to be able to obtain an analytical solution representing the eigenfield \( u_{lp} \), unfortunately this is not the case since the radial extent of the aperture in the upper limit is finite, hence the use of numerical techniques is unavoidable. Two general methods for the solution of Eq. (4.17) have been employed. The first one is a matrix method which consists on converting the round-trip integral equation into a matrix eigenvalue equation [43–45]. This method makes use of a Gaussian-Legendre quadrature rule and has some important advantages, namely it extracts the lowest \( N \) modes and eigenvalues at the same time, its accuracy is determined by the size \( N \) of the matrix [13], and it can also distinguish between nearly degenerate eigenvalues, which is a very difficult task when using iterative methods. The second method is the classical Fox-Li iterative scheme [3], which extracts the dominant or lowest-loss eigenmode, but as stated before it is not useful for discriminating between two modes with almost degenerate eigenvalues. The Fox-Li method has the advantage that, for cylindrical symmetries, the kernels in Eqs. (4.13) and (4.14) resemble a Hankel transform and, as a consequence, it is possible to use fast algorithms for propagating each radial eigenmode and eigenvalue in the resonator [46].

5.1 Reduction of the integral equation to a matrix equation

Transformation of Eq. (4.17) into a matrix equation has become a standard procedure for bare resonator analysis. Notice for instance that Eq. (4.17) can be approximated by a summation of the form

\[
\gamma u_{lp}(r_m) = \sum_{n=1}^{N} H(r_m, r_n) W_{nn} u_{lp}(r_n),
\]  

(5.1)
where $H(r_m,r_n)$ are the values of the kernel at $(r_m,r_n)$ and $W_{nn}$ represents a set of $N$ weighting factors, whose values in general depends on the numerical integration technique being employed. For our particular case, the weighting factors correspond to those of a Gauss-Legendre quadrature integration (refer to Appendix A for a detailed description of the quadrature method).

Typically the value of the radial coordinate is bounded by a maximum value where the input and output fields are small enough to be neglected, otherwise it is bounded by the radius $a_1$ of the double axicon. Let $r = [r_1,r_2,\cdots,r_N]$ and $w = [w_1,w_2,\cdots,w_N]$ be the abscissas and weight factors for the GL quadrature in the range $(0,a_1)$. If $u_{lp} = [u_1,u_2,\cdots,u_N]$ is a column vector representing the eigenfield evaluated at radius $r$, then from Eq. (5.1) the self consistency integral equation in (4.17) takes the matrix form

$$\gamma u_{lp} = (H * W) * u_{lp}, \quad (5.2)$$

where $H$ is a $N \times N$ matrix with elements $H_{m,n} = H(r_m,r_n)$, and $W$ is a diagonal matrix with elements $[w_1,w_2,\cdots,w_N]$.

The matrix product $H * W$ corresponds to the propagation matrix from plane 1 to plane 3. Since the round-trip propagation has been broken up by performing two plane to plane propagations, this product can be seen as the product of two propagation matrices, $P_{12}$ and $P_{23}$, corresponding to the equivalent matrix representation for the propagation integrals (4.13) and (4.14) respectively. Eq.(5.2) can be rewritten as

$$\gamma u_{lp} = (P_{23} * P_{12}) * u_{lp}, \quad (5.3)$$

where $P_{m,n} = H_{m,n} * W$. Finally using this last representation of Eq.(4.17), the optical resonator eigenmodes and eigenvalues can be numerically extracted by finding the eigenvalues and eigenvectors of Eq.(5.3), that can be easily determined using the well known matrix eigenvalue algorithms.
5.2 The Fox-Li algorithm

Among the iterative methods for solving Eq. (4.17) the first and still most popular is that of Fox and Li [3]. Their numerical approach consists on the iterative simulation of round trips of an initial distribution \( u^{(0)}(r) \) in the resonator. In each round trip the distribution is propagated by solving Eq. (4.10), which for cylindrical symmetries resemble a Hankel transform as it was previously mentioned, a complete description of the calculation of Hankel transforms is described in the Appendix B. The propagation of the field distribution is continued until convergence is achieved.

5.2.1 Lowest-order eigenmode

Suppose that an initial field distribution \( u^{(0)}(r) \) is launched in the resonator, and that it can be expanded into a combination of its transverse eigenmodes as follows

\[
 u^{(0)}(r) = \sum_{lp} c_{lp} u_{lp}(r) \tag{5.4}
\]

where \( c_{lp} \) are the expansion coefficients. After each round trip, each mode component \( u_{lp} \) of the starting field is multiplied by its corresponding eigenvalue \( \gamma_{lp} \), thus after \( k \) successive round trips the field in Eq. (5.4) at the same reference plane will be

\[
 u^{(0)}(r) = \sum_{lp} c_{lp} \gamma_{lp}^{k} u_{lp}(r) \tag{5.5}
\]

Eq. (5.5) means that, after \( k \) round trips, the amplitude of each mode component will be attenuated by a factor of \( |\gamma_{lp}|^k \), thus if the largest eigenvalue is labeled with indices \( lp = 01 \), then the mode presenting the lowest attenuation (lowest loss) will be \( u_{01} \), and each other eigenmode will suffer larger losses, therefore they will die out at faster rate, so that the mode \( 01 \) becomes dominant.

The waveform of the resonator eigenmodes \( u_{lp}(r) \) is initially unknown, thus the starting field \( u^{(0)}(r) \) cannot be represented by a combination of eigenmodes. However, after a sufficiently large number of bounces, the initially launched distribution should converge to the lowest loss eigenmode, even in the extreme case that the resonator posses only one resonating non-zero eigenmode. On the other side, if two eigenvalues have equal or nearly equal
magnitudes (mode degeneracy), the numerical calculation could not converge. The latter is a great disadvantage of the Fox-Li method, fortunately this case can be handled with the matrix method.

A numerical value for $\gamma_{01}$ is calculated in the limit

$$\gamma_{01} = \lim_{x \to \infty} \frac{u^{k+1}(r)}{u^k(r)}$$

(5.6)

As can be expected, for bare resonator calculations the field amplitude of $u^{(k)}$ decreases steadily due to diffraction losses, but this problem is solved by rescaling the overall field distribution, which is mathematically equivalent to the function performed by the gain medium.

Finally, when the steady state is reached the field $u^{(k)}(r)$ and its associated eigenvalue are simply the lowest loss eigenmode and eigenvalue respectively. The required number of round trips for the process to converge strongly depends on the initial field distribution, and is usually around $k = 250$.

The results presented in the following chapter are mainly based on the matrix method described here, however the Fox-Li approach has been employed to make appropriate comparisons and corroborate results, although there are some sections whose results were only obtained by means of the Fox-Li method.
Chapter 6

Results and Discussion

Once the theoretical framework for the analysis of the UBR has been extensively described, it turns out to be important to present the numerical results. In this chapter, the previously outlined methods to solve the self consistency integral equation are employed to extract the resonating modes of the cavity. Special attention is directed to the analysis of the diffraction losses behavior due to the variation of several parameters, such as the cavity length, the output coupler’s radius of curvature, and the characteristic angle of the axicon.

6.1 Resonating Modes

The physical parameters for the resonator used in the calculations correspond to typical values of a gas-discharge, fast axial flow cw-CO$_2$ laser resonator. A refractive axicon, whose base is perfectly reflective, is employed with an aperture size $a_1 = 3/8$ in $\simeq 10$ mm, index of refraction $n = 2.4$ (corresponding to commercially available zinc selenide ZnSe axicons), and wedge angle $\alpha = 0.5^\circ$; the output mirror’s aperture is $a_2 = a_1/2 = 5$ mm, as it will be shown later in this chapter this aperture value minimizes diffraction losses, although it can be arbitrarily varied. Finally, the wavelength of the emitted radiation is $\lambda = 10.6 \, \mu m$. From Eq. (3.1) the values for the cavity length and apex angle are calculated, yielding $L = 40.92$ cm, and $\theta_0 = 12.22 \times 10^{-3}$ rad. A 200 point Gauss-Legendre quadrature was used to solve Eq. (5.3).

In order to make appropriate comparisons between the theoretical Bessel beams and the finite approximations produced by the resonator, the fundamental mode $(l, p) = (0, 1)$ of the UBR with plane output mirror (i.e. $R \rightarrow \infty$) is considered first. Figure 6.1 shows the amplitude and phase of the transverse profiles at both the axicon plane ($RP_1$) and output
mirror \((RP_2)\).

![Graphs showing transverse profiles of the magnitude and phase of the fundamental Bessel mode at the axicon and output planes.](image)

Figure 6.1: Transverse profiles of the magnitude and phase of the fundamental Bessel mode at (a), (b) the axicon plane, and (c), (d) the output plane, for a resonator with output flat mirror and \(a_2 = a_1/2\). Dotted lines show results for the theoretical \(J_0\) beam.

The eigenfield and the eigenvalue at the axicon plane were determined by using the matrix method described in section 5.1. As stated in chapter 3, the field at the output mirror corresponds to an ideal zero-th order Bessel beam \(J_0(k\rho)\) modulated by a bell shaped amplitude function. The comparison between the actual eigenfield and the theoretical Bessel beam shape is also shown in Fig. 6.1(c), notice that the field extends beyond the value of \(r = 5\) mm, the reason is that its magnitude is being calculated and plotted just before reflection on the output coupler, in agreement with the definition of the reference plane \(RP_2\) shown in Fig. 3.2. Figure 6.1(b) shows that the phase of the field at the axicon plane is almost linear, with slope \(k\theta_0\); this fact confirms that the field at the axicon behaves as a conical wave.

Consider now the case of the UBR, where the plane output mirror has been replaced by a convex spherical mirror with radius of curvature \(R = -50L\). The amplitude and phase of the fundamental mode at both the axicon plane \((RP_1)\) and just before the output mirror \((RP_2)\) are depicted in Fig. 6.2. Despite the appearance of a radial modulation produced by
the unstable configuration, the output pattern closely resembles a Bessel beam whose rings
preserve the same separation as in the case shown in Fig. 6.1 for $R \to \infty$. This result is in
agreement with the fact that the ring separation is determined by the characteristic angle of
the conical mirror only, namely $k_t = k \sin \theta_0 \approx k \theta_0$, thus the separation between consecutive
Bessel fringes is $\theta_0/2\lambda$. As expected, the phase of the mode just before the axicon plane
behaves linearly according to $k \theta_0 r$. Additionally, the behavior of the fundamental Bessel
mode is presented in Fig. 6.3 for different values of the axicon wedge angle $\alpha$, and the same
radius of curvature $R = -50L$, it is evident that the radial separation of the Bessel fringes
increase as the value of the angle becomes larger, this is also an expected result, since in the
limit $\alpha \to 0$, the effect of the axicon ceases, and it behaves just as a plane mirror.

![Figure 6.2: Transverse profiles of the magnitude and phase of the fundamental Bessel mode at (a), (b) the axicon plane, and (c), (d) the output plane, for an UBR with $R = -50L$ and $a_2 = a_1/2$. Dotted lines show results for the theoretical $J_0$ beam.](image)

It has been pointed out that higher-order solutions $u_{lp}$ and $\gamma_{lp}$ for Eq. (4.17) can also be
obtained, in fact these solutions represent higher-order resonating modes of the cavity and
experience the same modulation than the fundamental mode. For instance, when the output
coupler is plane, the ideal transverse field is given by Eq. (3.2), but for a convex spherical
output coupler there is no analytical solution. Figure 6.4 shows the magnitude and phase of the eigenfield corresponding to the second order \((l, p) = (2, 1)\) for a radius of curvature \(R = -50L\) at the axicon and output planes.

Figure 6.3: Magnitude of transverse profiles of fundamental Bessel modes, for different values of the wedge angle \(\alpha\) of the axicon at (a) the axicon plane (b) the output mirror. \(R=-50L\).

It is observed again in Fig. 6.4(b) that the phase of the eigenfield at the axicon plane is linear. On the other hand, Fig. 6.4(c) shows excellent agreement between the numerical and theoretical results, this similarity also remains for the mode \((l, p) = (3, 1)\), corresponding to a \(J_3\) beam. However, for higher angular orders, the modes resemble a \(J_l\) field only near the center of the output mirror, as shown in Fig. 6.5, and there exists a considerable amount of light near the edges of the mirror, a large fraction of this light will be lost due to diffraction
Figure 6.4: Transverse profiles of the magnitude and phase of the second-order Bessel mode at (a), (b) the axicon plane, and (c), (d) the output plane, for an UBR with $R = -50L$ and $a_2 = a_1/2$.

hensive to the finite extent of the output mirror. The effect of the finite aperture can also be seen in the phase distortion near the edges of the output coupler in figures (6.1)-(6.4).

In general, it can be noticed that there exists very little difference between the eigenmodes of the resonator and theoretical Bessel-beams, however the similarity is stronger near the center of the aperture, since the field is attenuated at the edges. Apart from these facts, the UBR also offers better modal discrimination between higher order modes than the SBR, which will be discussed in depth later in this chapter.

### 6.2 Two-dimensional field distributions

In order to obtain a full picture of the transverse properties of the UBR eigenmodes, their two-dimensional distribution is obtained by means of the Fox-Li method described in section 5.2, but now propagation is performed in the Fourier space as described in Appendix C. Figure 6.6 displays the distributions for the first four angular modes ($l = 0, \ldots, 3$), for visualization
purposes the transverse spatial frequency of the modes was modified by reducing the value of the axicon wedge angle from 0.5° to 0.3°. The one-dimensional profile corresponding to each mode is also shown on the lower plots, although a stronger modulation is present on the transverse output profiles, excellent agreement exists between the four modes and their corresponding theoretical Bessel beams.

An important parameter that deserves mention is the Fresnel number of the cavity $N_F$. For the UBR shown in Fig. 3.1 it is defined by

$$N_F = \frac{a_1^2}{L\lambda} = \frac{2a_1\theta_0}{\lambda}$$  \hspace{1cm} (6.1)$$

where $\lambda$ is the free-space wavelength, and Eq. (3.1) was used to eliminate $L$. The physical interpretation of $N_F$ is that it corresponds to the number of Fresnel zones for the fundamental mode across the output mirror, as seen from the center of the opposite mirror [29], therefore Eq. (6.1) tells us that the number of Bessel fringes at the output plane is only defined by the axicon parameters. Taking the simulation parameters employed in Fig. 6.6, an outer Fresnel number $N_F = 13.83$ was calculated for the aperture range $[-5,5]$, obtaining an excellent agreement with the number of Fresnel ripples shown in Fig. 6.6(e) for the fundamental mode. If one is particularly interested in generating a Bessel beam with a predefined number of rings, then it is possible to fix a $N_F$, and manipulate Eq. (6.1) to determine the aperture radius and characteristic angle of the axicon that will satisfy the particular choice.

Referring to Fig. 6.6, a very special characteristic of Bessel beams can be outlined, with the exception of the zeroth order mode, the rest of the higher modes present a phase...
singularity, or vortex, at the origin. This singularity is confined to the central dark region, which is in fact more sharply defined than the bright spot at the center of the $J_0$ beam. Although the misalignment effects of the axicon and output mirror are not considered in this research, it is important to mention that the alignment of both elements is crucial, especially for higher angular orders, since their vortices tend to break apart into $l$ single vortices [32].

Figure 6.6: (a) Two-dimensional distribution of $|U|$, and transverse profile at the output mirror plane for the modes (a) $l = 0, p = 1$, (b) $l = 1, p = 1$, (c) $l = 2, p = 1$, (d) $l = 3, p = 1$. $R = -50L$, $\alpha = 0.3^\circ$, and $a_2 = a_1/2$.

6.3 Intracavity field distribution

The classical Fox and Li iteration method was implemented to determine numerically the passive three-dimensional field structure of the cavity modes. For this purpose, the diffractive field calculations are based on the angular spectrum of the plane waves representation utilizing the Fast Fourier Transform (FFT) algorithm. The transverse field is sampled in the reference plane over a grid of $512 \times 512$ points. Typically around 250 round-trips are required for the process to converge, starting from an arbitrary field distribution. The three-dimensional intracavity field distribution was obtained by calculating the field at 200 transverse planes evenly spaced through the unfolded cavity.

The intracavity field distribution of the dominant lowest-order mode in the UBR, with
Figure 6.7: Passive three-dimensional intracavity field distributions of the UBR for (a) \( R = -25L \), (b) \( R = -50L \), (c) \( R = -75L \) and (d) \( R = -100L \). The eigenfield is first forward propagated from the axicon plane to the output plane, and returned backwards to the axicon plane.

convex output mirror of different curvature radius, is presented in Fig. 6.7. Forward propagation goes from the axicon plane at \( z = 0 \) to the output mirror plane at \( z/L = 1 \). Reverse propagation goes from the output mirror at \( z/L = 1 \) to the axicon plane at \( z/L = 2 \). The axicon and the output mirror extend in transverse dimension from \(-1\) to \(1\) and from \(-0.5\) to \(0.5\) in normalized units \( r/a_1 \), respectively. The corresponding transverse fields at the axicon and output mirror planes for the value of \( R = -50L \) were already calculated through the eigenequation (4.17) and depicted in Fig. 6.2.

The results shown in Fig. 6.7 clearly illustrate the conical nature of the field within the cavity. At middle plane of the axicon the field is approximately a plane wave, after crossing
the axicon, the diffraction pattern of the conical wave exhibits a bright line focus surrounded by a series of cylindrical concentric side lobes with gradually diminishing intensity. A large number of Fox-Li simulations were performed for a variety of initial conditions, including uniform plane waves, gaussian profiles with different widths, and random noisy transverse patterns. Regardless of the initial condition, the field always converged to the dominant mode of the cavity with the expected profile and radial frequency characteristics imposed by the geometry parameters.

It should be noticed that when the propagating field reaches the output mirror \((z/L = 1)\), the most exterior Fresnel ripples are eliminated due to the finite extent of the mirror, therefore only the fringes contained within the area of the output coupler survive. Despite of this truncation, is interesting to see how the light distribution reconfigures so that the shape of the initial distribution is recovered at the input plane. Another interesting fact is that the field at the axicon and output planes present a central peak which is preserved during almost all the propagation distance \(L\) with some slight variations, the latter is clearly seen in Figs. 6.7(b), (c), and (d), in contrast to the annular distribution that would be expected in the case of the SBR with concave output mirror.

### 6.4 Diffractive losses and frequency shifts

The relationship in Eq. (3.1) between the cavity parameters shown in Fig. 3.1 seems to be very restrictive. Hence, from a practical point of view, it is of great interest to study the effect of varying these parameters, as well as the radius of curvature of the output coupler, on the modal distribution and energy losses. The relevant output characteristics are the transverse field profile, the diffractive loss, and the resonant frequency shift.

#### 6.4.1 Variation of the radius of curvature of the output coupler

The effect of varying the radius of curvature of the output mirror on the field distribution at the axicon and output planes is shown in Fig. 6.8. For \(R = -200L\) there exists very little differences between the fundamental eigenmode of the resonator and the ideal Bessel beam shown in Fig. 6.1(c). As \(R\) decreases, the transverse field at the axicon plane tends to concentrate around the vertex of the conical mirror, and the Bessel rings are modulated by a
radial amplitude factor. Note that the ring separation remains constant under the variation of $R$. The output radial distribution of the $J_2$ Bessel mode for several radii of curvature is depicted in Fig. 6.9. One can see that the calculated eigenfields are multi-ringed, and that as $R$ increases the pattern becomes more similar to a theoretical second-order Bessel beam. Note that higher-order modes experience the same radial modulation than the fundamental mode.

Figure 6.8: Transverse profiles of the fundamental Bessel modes magnitudes at (a) the axicon plane, and (b) the output mirror’s plane for several ratios $R/L$, with $\alpha = 0.5^{\circ}$. 
Figure 6.9: Transverse profiles of the second-order Bessel modes magnitudes at (a) the axicon plane, and (b) the output mirror’s plane for several ratios $R/L$, with $\alpha = 0.5^\circ$. 
6.4.2 Diffractive losses in terms of the varying radius

The loss behavior corresponding to the lower-order modes resonating within the UBR is now considered. The loss $\Gamma_{01}$ for the fundamental mode, given by equation 4.19, is depicted in Fig. 6.10(a) as a function of the normalized radius $R/L$ for two different values of the axicon wedge angle $\alpha$ and for two different values of the output mirror radius $a_2$. These plots were computed by finding the eigenvalues from Eq. (4.17) for a large number of radii of curvature in the range $-200 < R/L < -30$, and also corroborated by means of the Fox-Li method [markers $\circ$ in Fig. 6.10(b)] using the fast Hankel transform algorithm described in reference [46], which is employed because it considerably reduces the computation time that would be required by the FFT algorithm.

![Figure 6.10](image)

Figure 6.10: (a) Diffractive losses $\Gamma = 1 - |\gamma|^2$ in terms of the normalized radius $R/L$ for the fundamental mode, with different wedge angles and different aperture sizes of the output mirror. (b) Comparison of results with the Matrix (thin line) and Fox-Li ($\circ$ marker) methods.

The diffractive losses curves for higher order modes also exhibit the same behavior as the curve for the fundamental mode, as shown in Fig. 6.11(a). There are some conclusions that can be inferred from the loss curves in Figs. 6.10 and 6.11(a): in general, the mode loss increases as the output mirror becomes more and more convex, loss increases as the aperture diameter of the output mirror increases, and loss increases as the axicon wedge angle decreases. It has been found numerically that the lowest loss curve occurs when the value of the output aperture is $a_2 \simeq a_1/2$.

Additionally, it is expected that the resonant frequency of a particular resonating mode in the UBR, given by Eq. (4.21), will change if the value of $R$ changes. The new resonant frequency can be written as $\nu = \nu_\infty + \Delta \nu$, where $\nu_\infty$ is the resonance frequency of the resonator.
Figure 6.11: (a) Diffractive losses $\Gamma = 1 - |\gamma|^2$ and (b) relative phase shift behavior $\Delta \beta/\pi$ in terms of the normalized radius $R/L$ for the modes ($l = 0 \ldots 2, p = 1$) with plane output mirror, and $\Delta \nu$ the frequency shift introduced by the variation of the radius of curvature. The relative phase shift experienced by a resonating mode inside the cavity is defined as $\Delta \beta = \beta(R) - \beta_\infty$, with $\beta(R)$ being the phase angle of the corresponding eigenvalue in terms of $R$, and $\beta_\infty$ the angle for the case when $R \to \infty$, i.e. plane output mirror.

By a simple manipulation of Eq. (4.21), the frequency shift $\Delta \nu$ in terms of the phase shift $\Delta \beta$ is found to be

$$\Delta \nu = -\nu_0(\Delta \beta/\pi),$$

hence, when $\Delta \beta > 0$ the resonant frequency $\nu$ has a smaller value than $\nu_\infty$, on the other hand, when $\Delta \beta < 0$ the resonant frequency $\nu$ is greater than $\nu_\infty$. Figure 9(b) illustrates the normalized phase shift $\Delta \beta/\pi$ for the first four azimuthal modes $l = (0, 1, 2)$ and $p = 1$. The curves for the relative phase shift decrease asymptotically to 0, which is the limiting value corresponding to the UBR with plane output coupler. It is readily evident, from these curves, that the resonant frequency of the resonating modes in the UBR is greater than the resonant frequency for the cavity with plane output mirror, besides this frequency decreases as the output coupler becomes more convex.

### 6.4.3 Diffractive losses due to cavity length variations

To study the effect of varying the cavity length the parameter $\mu = L/L_0$ is defined, where $L_0$ is the unchanged cavity length in Eq. (3.1) and $L$ is the current length. The fundamental mode patterns at the axicon and output mirror planes are depicted in Fig. 6.12 for $\mu = [0.8, 1.0, 1.2]$. Note that the radial separation of the Bessel rings remains practically
constant, which means that the transverse component of the propagation vector is not affected by a change in the cavity length.

Figure 6.12: Transverse field patterns at the output and axicon planes corresponding to the length factors $\mu = 0.8, 1$ and $1.2$, for convex output mirror with $R = -50L$ and $\alpha = 0.5^\circ$.

In Fig. 6.13 the profiles for $\mu = 0.5$ and $\mu = 1.5$ are also shown, again the radial separation between Bessel fringes is preserved with respect to the profile for the unchanged cavity. An interesting fact is that, despite the large differences between the transverse profiles for the values of $\mu = 0.5, 0.8, 1.2, 1.5$ at the axicon plane, their distributions at the output plane are very similar, however it can be seen in Figs. 6.12 and 6.13 that these profiles experience a different modulation. If the value of the cavity length is diminished towards zero, the shape of the output transverse profile starts to become irregular, in the sense that it does not resemble any familiar shape, this occurs because in such situation the resonator becomes highly unstable, and although Eq. (4.17) may have a solution for $L$ very close to zero, the resonator does not longer represent a real or meaningful system. On the other hand, if the length of the cavity is further increased, the effect on the transverse profile starts to become important. The radial frequency of the Bessel fringes decreases and there is more light near the edges of the output coupler, consequently the diffraction losses increase as the length increases.

The main difference between transverse profiles for different cavity lengths is related to diffraction losses, it is observed that the modes for $\mu \neq 1$ present larger losses than the mode for $\mu = 1$, therefore it is expected that the amplitudes of their output profiles decrease faster. The curve for the losses is depicted in Fig. 6.14(a) for the first two azimuthal modes ($l = 0, 1, p = 1$) and $R = -50L$. As in the case reported for the stable Bessel-Gauss resonator
Figure 6.13: Transverse field patterns at the output and axicon planes corresponding to the length factors $\mu = 0.5, 1$ and $1.5$, for convex output mirror with $R = -50L$ and $\alpha = 0.5^\circ$.

(SBR) [13], it is remarkable that the minimum for the loss curves does not occur at the value of $\mu = 1$, as expected from the geometrical analysis, but at a slightly different value of $\mu$. For the fundamental mode $(0,1)$ it occurs at $\mu \simeq 0.966$, whereas for the mode $(1,1)$ about 1.058. Notice also that this result is a consequence of the wave nature of the fields, and cannot be predicted by geometrical optics. Additionally, the mode $(1,1)$ becomes the fundamental mode of the resonator as the cavity length increases, this mode crossing due to the varying cavity length also occurs between other higher order modes.

Figure 6.14: Loss and resonant frequency shift behavior as a function of the length factor $\mu$ for the first two angular modes $(l,p) = (0,1)$ and $(1,1)$, with $R = -50L$ and $\alpha = 0.5^\circ$.

Finally, figure 6.14(b) depicts the relative phase shift behavior for $l = 0$ and 1. The relative phase shift has been redefined in terms of the length $L$ as $\Delta \beta = \beta(L) - \beta_{L_0}$, where $\beta(L)$ is the phase angle of the eigenvalue as a function of the varying length, and $\beta_{L_0}$ is the angle for the unchanged cavity length given by Eq. (3.1). It can be readily seen that the relative phase shift behavior is almost linear, and that it decreases for an increasing $\mu$. 47
6.4.4 Diffractive losses in terms of the output mirror’s varying aperture

The majority of the results presented in this research were obtained employing an output mirror aperture equal to $a_1/2$, it was argued without proof that this value reduces diffraction losses. No analytical proof is given to support the latter argument, instead a numerical analysis revealed that the diffraction losses increase when $a_2$ moves away from this value. The plot in Fig. 6.15 shows the behavior of the losses as a function of the varying aperture of the output mirror, although the shape of the curve does not exhibit a smooth variation, it can be seen that the region of minimal losses for the fundamental mode ($l = 0, p = 1$) is very close to the value $a_2 = a_1/2$, whereas for the mode ($l = 1, p = 1$) is approximately equal to $0.45a_1$, thus the use of $a_2 = a_1/2$ is justified under the requirement that diffraction losses be minimized.

![Figure 6.15: Diffractive losses $\Gamma = 1 - |\gamma_{lp}|^2$ for the modes ($l=0,1;p=1$) in terms of the normalized aperture ratio $a_2/a_1$. Diffractive losses for the fundamental mode are minimized when $a_2 \simeq a_1/2$.](image)

6.5 Transition between the stable and unstable Bessel resonators

In order to complete the analysis of the UBR, a comparison with its equivalent stable counterpart (SBR) is performed in this section. The analysis is mainly focused on the diffractive losses behavior for both resonators. The configuration considered for the UBR cor-
responds to that shown in Fig. 3.1, but now considering equal apertures for both the axicon and output mirrors. The equivalent stable cavity consists of a similar array, with a concave spherical output coupler instead of a convex one. Simulation parameters for this section are summarized in Table 6.1.

Table 6.1: Values for the numerical analysis of the UBR and SBR

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axicon aperture</td>
<td>$a_1 = 3/8\text{in} \approx 10\text{mm}$</td>
</tr>
<tr>
<td>Output aperture</td>
<td>$a_2 = a_1$</td>
</tr>
<tr>
<td>Index of refraction</td>
<td>$n = 2.4$</td>
</tr>
<tr>
<td>Wedge angle</td>
<td>$\alpha = 0.5^\circ$</td>
</tr>
<tr>
<td>Cavity length</td>
<td>$L = 40.92\text{cm}$</td>
</tr>
<tr>
<td>Apex angle</td>
<td>$\theta_0 = 12.22 \times 10^{-3}\text{rad}$</td>
</tr>
</tbody>
</table>

As is usual in the analysis of optical resonators, the eigenmodes of Eq. (4.17) are sorted in order of decreasing magnitude of its corresponding eigenvalue as $|\gamma_1| \geq |\gamma_2| \geq \ldots$, or which is equivalent in order of increasing mode losses. Numerical values are included in Table 1 for the first ten modes of the UBR ($R = -50L$), the SBR ($R = 50L$), and the resonator with plane output mirror ($R \to \infty$).

Table 6.2: Diffractive losses per pass for the first 10 modes

<table>
<thead>
<tr>
<th>$R = -50L$</th>
<th>$R \to \infty$</th>
<th>$R = 50L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode</td>
<td>Loss</td>
<td>Mode</td>
</tr>
<tr>
<td>0,1</td>
<td>0.11346</td>
<td>0,1</td>
</tr>
<tr>
<td>1,1</td>
<td>0.16585</td>
<td>1,1</td>
</tr>
<tr>
<td>0,2</td>
<td>0.16826</td>
<td>2,1</td>
</tr>
<tr>
<td>0,3</td>
<td>0.21633</td>
<td>0,2</td>
</tr>
<tr>
<td>1,2</td>
<td>0.24841</td>
<td>1,2</td>
</tr>
<tr>
<td>2,1</td>
<td>0.25777</td>
<td>3,1</td>
</tr>
<tr>
<td>1,3</td>
<td>0.28333</td>
<td>0,3</td>
</tr>
<tr>
<td>5,1</td>
<td>0.28878</td>
<td>2,2</td>
</tr>
<tr>
<td>6,1</td>
<td>0.29033</td>
<td>4,1</td>
</tr>
<tr>
<td>4,1</td>
<td>0.29376</td>
<td>1,3</td>
</tr>
</tbody>
</table>

In Fig. 6.16, the eigenvalue spectra of the UBR and the SBR are shown as a function of the normalized radius of curvature $R/L$. The plots in Fig. 6.16 clearly illustrate that the transition between the UBR and the SBR is characterized by a continuous change in the
associated diffraction losses and frequency shifts. These results were determined from matrix
equation (5.3). The loss curves of all modes exhibit a monotonically increasing behavior as
the output mirror becomes more convex. This result is expected from the fact that in the
SBR the field is constantly refocused by the concave mirror and the axicon, whereas in the
UBR only the axicon forces the field to propagate towards the optical axis. Note that mode
crossing points are present for higher modes.

Figure 6.16: Transition between the stable and unstable regions of the Bessel resonator. The
(a) diffractive losses $\Gamma = 1 - |\gamma|^2$ and (b) normalized phase shifts $\Delta \beta / \pi$ are depicted as a
function of the normalized radius $R/L$. Mode crossings are present for higher-order modes.
In subplot (b) the phase curves for the considered modes in subplot (a) are contained within
the region between the phase curves for modes (0,1) and (0,4).

In Fig. 6.17, a comparison between the output field profiles of the SBR and the UBR is
performed for the $J_0$ and $J_1$ Bessel modes, dotted curves depict the ideal Bessel profiles. It
was mentioned before that resonating modes of the SBR retain a Gaussian radial modulation
on the Bessel rings, whereas the UBR exhibits a more uniform modulation, as Fig. 6.17
illustrates. The latter means that the modes produced by the UBR, with the appropriate
Figure 6.17: Comparison between output profiles of the UBR \((R = -50L)\) and the SBR \((R = 50L)\) for the fundamental and first-order Bessel modes. Dotted lines correspond to the ideal Bessel profiles.

parameters, are more similar to Bessel beams than those produced by the SBR, which actually correspond to Bessel-Gauss modes.
A detailed analysis of the resonating modes in the axicon-based unstable Bessel resonator with spherical convex output mirror has been presented. The mode behavior under the variation of the radius of curvature of the output coupler, the axicon angle, and the cavity length, taking into account the finite aperture size of the mirrors composing the cavity, has also been extensively studied.

7.1 General conclusions

The most important conclusions are summarized as follows:

- UBRs support higher-order Bessel-like modes, which satisfy a biorthogonal relation rather than an orthogonal relation.

- The eigenvalue matrix method, based on the discretization of the Huygens-Fresnel self-consistency integral equation, is particularly useful for extracting the first $N$ eigenfields and eigenvectors (i.e. losses and frequency shifts) of the resonating modes at the output coupler and the axicon plane of the UBR. The Fox-Li algorithm can be used to efficiently extract the dominant mode of the cavity. Excellent agreement between the matrix and Fox-Li method was achieved.

- The UBRs possess higher transverse modal discrimination in favor of the fundamental mode than SBRs.

- While the SBR retains a Gaussian radial modulation on the Bessel rings, the UBR exhibits a more uniform amplitude modulation that produces output profiles more similar to ideal Bessel beams. It now seems clear that for a laser system characterized by a
moderate gain per pass (>50% per pass), the best practical cavity for obtaining nearly ideal Bessel beams will be an UBR with $R \simeq -50L$.

- Given the light wavelength, the radial separation of the Bessel rings in the UBR depend only on the characteristic angle of the conical mirror.

- The transition between the UBR and the SBR is characterized by a continuous change in the associated diffraction losses and frequency shifts.

- In general, the mode loss increases as the output mirror becomes more convex, and as the axicon wedge angle decreases. The lowest loss curve occurs when the value of the aperture radius of the output coupler is $a_2 \simeq a_1/2$.

- The mode $(l, p) = (0, 1)$ exhibits the lowest loss curve for practically all values of the radius of curvature of the output mirror. However, mode crossing points are present for higher modes.

- The mode $(1, 1)$ becomes the fundamental mode of the resonator as the cavity length increases beyond $\sim 1.15$ times the unchanged cavity length defined by Eq. (3.1). Mode crossing points due to the varying cavity length also occur between other higher order modes.

- The cavity loss for the fundamental mode is minimized when the cavity length is about the 96% of the value predicted by geometrical optics.

- The reduction of the aperture ratio $a_2/a_1$, from 1 to 1/2 in favor of the output convex mirror, reduces edge-diffraction effects and prevents significant energy spillage across the vertex of the axicon mirror.

The analysis presented in this thesis consolidates previous works on the production of Bessel and Bessel-Gauss beams in laser resonators [13, 14, 27, 28], and introduces a formal methodology for the analysis of unstable Bessel resonators. Furthermore it studies for the first time the modal distribution of higher order Bessel beams resonating within a UBR.
7.2 Future work

There is still work to be done in order to achieve a full understanding of the UBR properties. For instance, work concerning the effect of the axicon and output mirror misalignments is being performed. The analysis of the resonator considering the effect of the gain medium on the three dimensional mode distribution is also in progress. Additionally the $M^2$ factor of the UBR modes needs to be calculated. Finally, a formal exploration of the polarization of the output beams seems to be promising.
Appendix A

Gaussian quadrature method

In general, gaussian quadrature methods for integration are based on the assumption that the integrand can be represented as the product between a weighting function and a polynomial, namely if \( g(x) \) is the integrand function, then \( g(x) = W(x)f(x) \). The function \( W(x) \) can be chosen so that singularities can be removed from the desired integral, then for a given function \( W(x) \) and an integer \( N \), gaussian quadrature seeks for a set of weights \( w_i \) and abscissas \( x_i \) such that the approximation

\[
\int_{a}^{b} W(x)f(x)dx = \sum_{i=1}^{N} w_i f(x_i) \quad (A.1)
\]

is exact if \( f(x) \) is a polynomial [48]. The fundamental theorem of Gaussian quadrature states that the optimal abscissas of the \( N \)-point quadrature are precisely the \( N \) roots of an orthogonal polynomial \( p(x) \), for the same interval and weighting function \( W(x) \), satisfying

\[
\int_{a}^{b} p(x)W(x)x^kdx = 0, \quad (A.2)
\]

where \( k = 0, 1, \ldots, N-1 \). Gaussian quadrature is optimal because it fits all polynomials up to degree \(< 2N \) exactly. For the particular case of a weighting function \( W(x) = 1 \), the abscissas correspond to the roots of the \( N \)-th order Legendre Polynomial \( P_N(x) \) in the interval \([-1, 1]\). Hence Eq. (A.1) reduces to

\[
\int_{a}^{b} f(x)dx = \sum_{i=1}^{N} w_i f(x_i) \quad (A.3)
\]

A change of variable is necessary to change the integration interval of Eq. (A.3) from \([a,b]\) to \([-1,1]\), which is required in order to satisfy the fundamental theorem of gaussian quadrature
for \( p(x) = P_N(x) \) and \( W(x) = 1 \). The integral can be transformed as

\[
\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^1 f(z)dz = \frac{b-a}{2} \sum_{i=1}^N w_i f(z_i) \tag{A.4}
\]

using the change of variable \( x = \frac{1}{2}(b + a) + \frac{1}{2}(b - a)z \).

Finally, the zeros \( z_i \) for the Legendre polynomials can be found in tables, or numerically by employing Newton’s method with deflation, whereas the weighting factors are calculated as [49]

\[
w_i = \frac{2}{(1 - x_i^2)[P'_N(x_i)]^2} = \frac{2(1 - x_i^2)}{(n + 1)^2[P'_{N+1}(x_i)]^2} \tag{A.5}
\]

The implementation of Eq. (A.4) is easy, and can be reduced to a matrix multiplication of the form \( F_NW \), where \( F_N \) is a row vector given by \( [f(z_1), f(z_2), \ldots, f(z_N)] \) and \( W \) is a column vector, with \( [w_1, w_2, \ldots, w_N] \) begin the weighting factors of the Gauss-Legendre quadrature method.
Appendix B

Hankel Transform

A function of two independent variables is said to be separable in one coordinate system, if for that particular system it can be written as the product of two functions, each of which only depends on one of the two variables. Consider a function that is separable in polar coordinates \((r, \theta)\) as

\[ g(r, \theta) = g_R(r)g_\Theta(\theta), \quad (B.1) \]

if \(g(r, \theta)\) is assumed to be independent of the angular variable, then the function is said to be circularly symmetric, thus

\[ g(r, \theta) = g_R(r). \quad (B.2) \]

Functions of the form (B.2) are of special interest since many optical systems are circularly symmetric, therefore they can be described by such functions. The Fourier transform of \(g\) in a rectangular coordinate system \((x, y)\) is

\[ \mathcal{F}[g(x, y)](u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)e^{-2\pi i (ux + vy)} \, dx \, dy, \quad (B.3) \]

taking advantage of the circular symmetry of \(g\), the Fourier transform of Eq. (B.3) can be expressed in polar coordinates using the following change of variables for both \((x, y)\) and \((u, v)\),

\[
\begin{align*}
  x &= r \cos \theta & u &= \rho \cos \phi \\
  y &= r \sin \theta & v &= \rho \sin \phi \\
  r &= \sqrt{x^2 + y^2} & \rho &= \sqrt{u^2 + v^2}
\end{align*}
\]

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inserting the previous expressions in Eq. (B.3) and using Eq. (B.2), the Fourier transform for circular symmetric functions is given by

\[ f(\rho) = \int_0^\infty \int_0^{2\pi} g(r) e^{-2\pi i r \rho (\cos \theta \cos \phi + \sin \theta \sin \phi)} r dr d\phi \]

\[ = \int_0^\infty \int_0^{2\pi} g(r) e^{-2\pi i r \rho (\cos \phi - \theta)} r dr d\phi \]

\[ = \int_0^\infty \int_0^{2\pi} g(r) e^{-2\pi i r \rho \cos \phi} r dr d\phi \]

\[ = \int_0^\infty g(r) \left[ \int_0^{2\pi} e^{-2\pi i r \rho \cos \phi} d\phi \right] r dr \]

\[ = 2\pi \int_0^\infty g(r) J_0(2\pi r \rho) r dr, \quad \text{(B.4)} \]

where \( J_0(2\pi r \rho) \) is the zeroth order Bessel function of the first kind, with integral representation [51]

\[ J_0(2\pi r \rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{-2\pi i r \rho \cos \phi} d\phi \quad \text{(B.5)} \]

The expression in Eq. (B.4) is known as the zeroth order Hankel transform of \( g(r) \), and is also called the Fourier-Bessel transform. The Hankel transforms pair is

\[ f(\rho) = 2\pi \int_0^\infty g(r) J_0(2\pi r \rho) r dr \quad \text{(B.6)} \]

\[ g(r) = 2\pi \int_0^\infty f(\rho) J_0(2\pi r \rho) r dr \quad \text{(B.7)} \]

A fast, accurate, and reliable numerical method for the evaluation of Hankel transforms can be found in Ref. [46].
Appendix C

Fourier space propagation

Since the first description of the transverse mode structure for a stable resonator, made by Fox and Li in 1961 [3], based upon the Fresnel-Kirchhoff diffraction integral, there has been a growing interest in the analysis of the mode structures for particular cavities. This fact has favored the development of many algorithms to simulate the propagation of the optical wave field inside a resonator. Particularly, a general approach based on the angular spectrum of plane waves was described by Sziklas and Siegman [47], with the specific application to unstable resonators, however this method can also be applied to stable resonators, including the special types of novel resonators such as the UBR and SBR. The propagation algorithm based on the angular spectrum of plane waves works as follows.

Take the scalar Helmholtz equation

\[
(\nabla^2 + k^2) u(\vec{r}) = 0 \quad (C.1)
\]

Given the input scalar wave function \( u(r_T, z_0) \) at a transverse plane \( z = z_0 \), propagating in the positive \( z \) direction, the problem consists on determining the resultant scalar wave field \( u(r_T, z) \) at some later \( z > z_0 \).

On the transverse plane located at point \( z = z_0 \), the wave function \( u(r_T, z_0) = u(x, y, z_0) \) satisfies

\[
u(x, y, z_0) = \int \int_{-\infty}^{\infty} \tilde{u}(v_x, v_y, z_0) e^{i2\pi(v_x x + v_y y)} \, dv_x \, dv_y \quad (C.2)
\]

\[
\tilde{u}(v_x, v_y, z_0) = \int \int_{-\infty}^{\infty} u(x, y, z_0) e^{-i2\pi(v_x x + v_y y)} \, dx \, dy \quad (C.3)
\]
where \( \tilde{u}(v_x, v_y, z_0) \) represents the spatial frequency spectrum of the initial transverse disturbance. At a later plane, located at \( z = z_0 \), the propagated scalar wave \( u(x, y, z) \) and its frequency spectrum \( \tilde{u}(v_x, v_y, z) \) satisfy a similar two-dimensional Fourier transform pair,

\[
\begin{align*}
    u(x, y, z) &= \int \int_{-\infty}^{\infty} \tilde{u}(v_x, v_y, z) e^{i2\pi(v_x x + v_y y)} dv_x dv_y \quad (C.4) \\
    \tilde{u}(v_x, v_y, z) &= \int \int_{-\infty}^{\infty} u(x, y, z) e^{-i2\pi(v_x x + v_y y)} dx dy. \quad (C.5)
\end{align*}
\]

Each particular spatial frequency component in equation (C.2) corresponds to a plane wave propagating as

\[
e^{ik\hat{r} \cdot \hat{r}} = e^{i(2\pi v_x x + i2\pi v_y y + ik_z z)} \quad (C.6)
\]

where \( (2\pi v_x)^2 + (2\pi v_y)^2 + k_z^2 = k^2 = (w/c)^2 \), so that the \( z \) component of propagation is given (in the Fresnel approximation) by

\[
k_z(v_x, v_y) = \left[ k^2 - (2\pi v_x)^2 - (2\pi v_y)^2 \right]^{1/2} = k\left[ 1 - \frac{\lambda^2}{(v_x^2 + v_y^2)} \right]^{1/2} \quad (C.7)
\]

with \( k = 2\pi/\lambda \) being the wave number.

Since the propagated field in (C.4) must satisfy equation (C.1) with (C.2) as the boundary value, the propagated spectrum is related to the initial spectrum according to [50]

\[
\tilde{u}(v_x, v_y, z) = \tilde{u}(v_x, v_y, z_0) e^{ik_z \Delta z}, \quad (C.8)
\]

where \( \Delta z = z - z_0 \) is the propagation distance. The exponential term in the previous equation is commonly known as the propagator in Fourier space, therefore to obtain the resulting field after propagating an initial distribution \( u(x, y, z_0) \) over a distance \( \Delta z \), the spatial frequency spectrum of the initial field is just multiplied by \( e^{ik_z \Delta z} \). For the purposes of obtaining the mode structure of the resonator, propagation is performed back and forth within the resonator until convergence is acquired. In order to achieve accurate results, special consideration should be given to the sampling interval size for the required numerical Fourier transformation.
Bibliography


